MATHEMATICS · ~20 MIN · 3 SECTIONS · 5 QUESTIONS

De Morgan's Laws and Symmetric Difference v4

In this lesson

Apply De Morgan's laws to negate compound set expressions, compute symmetric differences, and solve equations involving symmetric difference.

Core idea

Complementing every component switches union and intersection, while symmetric difference keeps exactly the elements belonging to one set but not both.

1. Set Operation Notation
2. De Morgan Transformations
3. Symmetric Difference

1 Set Operation Notation

Term	Meaning	In the example
Union	Elements in at least one of the sets	$A \cup B = \{1, 2, 3, 4\}$
Intersection	Elements common to both sets	$A \cap B = \{2\}$
Difference	$A \setminus B$ contains elements in A but not in B; it is directional and equals $A \cap B^c$	$A \setminus B = \{1, 4\}$
Complement	Elements of the fixed universal set U that are not in the set	$A^c = \{3, 5\}$ when $U = \{1, 2, 3, 4, 5\}$

Set operations select elements according to membership: union means membership in at least one set, intersection means membership in both, difference means membership in the first set but not the second, and complement is always relative to one fixed universal set U. For example, if $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 4\}$ and $B = \{2, 3\}$, then $A \setminus B = \{1, 4\}$ and $A^c = \{3, 5\}$.

2 De Morgan Transformations

Key rule

Complement laws

$$(A \cup B)^c = A^c \cap B^c, \qquad (A \cap B)^c = A^c \cup B^c, \qquad (A \cup B \cup C)^c = A^c \cap$$

Apply the switch from the inside outward

For $U = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 3, 5\}$ and $B = \{2, 3, 6\}$:

$(A \cup B)^c = (\{1, 2, 3, 5, 6\})^c = \{4\}$

$A^c \cap B^c = \{2, 4, 6\} \cap \{1, 4, 5\} = \{4\}$

Therefore $(A \cup B)^c = A^c \cap B^c$; both sides give $\{4\}$.

For three sets, $(P \cap Q \cap R)^c = P^c \cup Q^c \cup R^c$: complement each operand and change $\cap$ t

A nested case simplifies from the innermost brackets: $((M \cup N)^c)^c = M \cup N$.

Quick practice

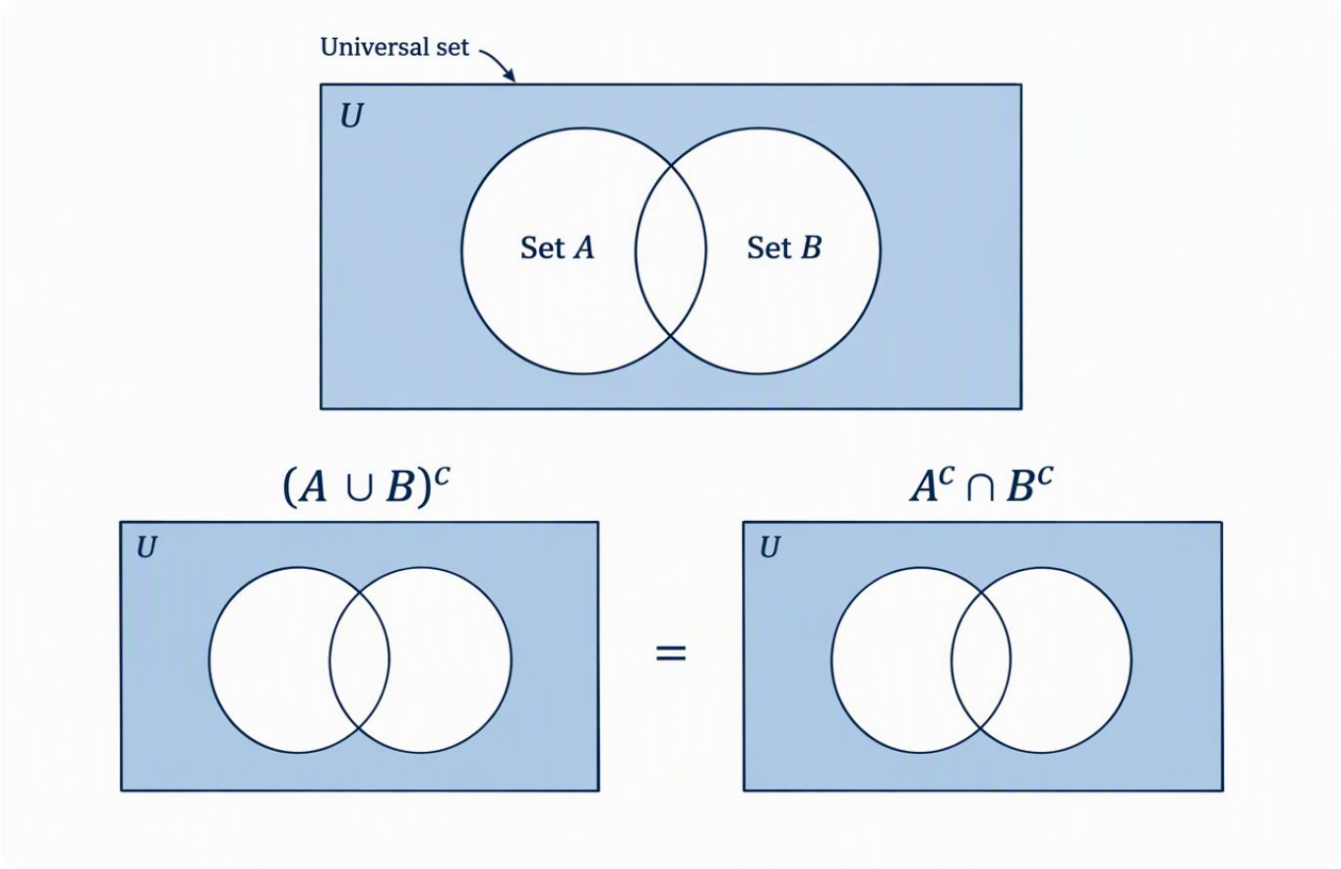
Practice

Let $U = \{1, 2, 3, 4, 5, 6, 7\}$, $P = \{1, 2, 5\}$, $Q = \{2, 3, 6\}$ and $R = \{4, 6, 7\}$. Which set equals $(P \cup Q \cup R)^c$?

Answer

$\varnothing$

Switch the three-set union to an intersection of complements, or observe that the union is all of U.



Outside the union means outside A and outside B.

3 Symmetric Difference

Key rule

The symmetric difference $A \Delta B$ contains elements that belong to exactly one of A and B, so elements in the overlap are excluded.

Equivalent descriptions and cancellation

$$A \Delta B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B), \qquad A \Delta A = \varnothing, \qquad A \Delta \varnothing = A, \qquad (X$$

Keep both exclusive parts

Let $C = \{1, 2, 4, 7\}$ and $D = \{2, 4, 5, 8\}$.

$C \setminus D = \{1, 7\}$

$D \setminus C = \{5, 8\}$

$C \,\Delta\, D = \{1, 7\} \cup \{5, 8\} = \{1, 5, 7, 8\}$

The equivalent form gives $(C \cup D) \setminus (C \cap D) = \{1, 2, 4, 5, 7, 8\} \setminus \{2, 4\} = \{1, 5, 7, 8\}$.

For $X \,\Delta\, \{3, 6\} = \{1, 3, 7\}$, apply $\Delta\{3, 6\}$ again; associativity and self-cancellation give X

$X = \{1, 6, 7\}$

Verification: $\{1, 6, 7\} \,\Delta\, \{3, 6\} = \{1, 3, 7\}$.

Special case	What happens	Example
A set with itself	Every element is shared, so nothing belongs to exactly one set	$\{2, 5\} \,\Delta\, \{2, 5\} = \varnothing$
A set with the empty set	No elements are added or removed	$\{1, 4, 6\} \,\Delta\, \varnothing = \{1, 4, 6\}$
One set contains the other	Only elements in the larger set but not the smaller remain	$\{1, 2, 3, 4\} \,\Delta\, \{2, 4\} = \{1, 3\}$

Quick practice

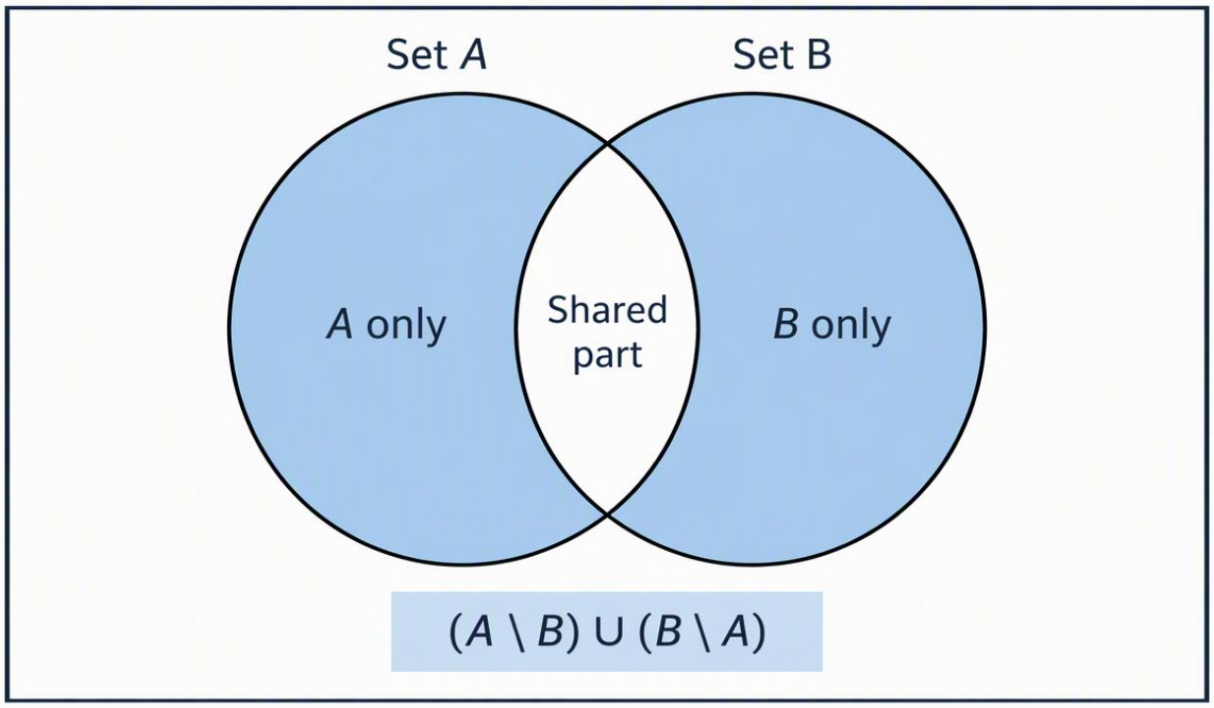
Practice

Which set X satisfies $X \,\Delta\, \{a, c\} = \{a, b, d\}$?

Answer

$\{b, c, d\}$

Apply symmetric difference with $\{a, c\}$ to both sides and cancel the repeated set.



Symmetric difference keeps the two non-overlapping parts and excludes the shared part.

Worked Example

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 3, 6\}$, $B = \{2, 3, 7\}$ and $C = \{3, 4, 8\}$. Which option gives both $(A \cup B)^c$ and $(A \cap B \cap C)^c$?

A $\{4, 5, 8\}$ and $\{1, 2, 4, 5, 6, 7, 8\}$

B $\{4, 5, 8\}$ and $\{3\}$

C $\{1, 2, 3, 6, 7\}$ and $\{1, 2, 4, 5, 6, 7, 8\}$

D $\{4, 5, 8\}$ and $\{1, 2, 4, 5, 6, 7\}$

E $\{1, 2, 3, 6, 7\}$ and $\{3\}$

Use the fixed universe, switch the connective while complementing every operand, and then read the resulting elements.

Form the union and remove it from U.

$A \cup B = \{1, 2, 3, 6, 7\}$, so $(A \cup B)^c = \{4, 5, 8\}$.

Apply the paired De Morgan transformation to the three-set intersection.

$(A \cap B \cap C)^c = A^c \cup B^c \cup C^c$.

Evaluate the intersection before complementing.

$A \cap B \cap C = \{3\}$, so $(A \cap B \cap C)^c = \{1, 2, 4, 5, 6, 7, 8\}$.

Select the option containing both results.

The pair is $\{4, 5, 8\}$ and $\{1, 2, 4, 5, 6, 7, 8\}$.

Answer

A — $\{4, 5, 8\}$ and $\{1, 2, 4, 5, 6, 7, 8\}$

The first result excludes every element in $A \cup B$, and the second excludes only the common element 3.

Worked Example 2 — Your Turn on the Marked Steps

Case Reverse-direction equation solved by cancellation, followed by substitution verification.

Let $A = \{2, 5, 7\}$ and $B = \{1, 5, 8\}$. Which set X satisfies $X \,\Delta\, A = B$?

A $\{1, 2, 7, 8\}$

B $\{1, 5, 8\}$

C $\{2, 5, 7, 8\}$

D $\{1, 2, 5, 7, 8\}$

E $\{1, 2, 8\}$

$\{1, 2, 5\}$

Apply ΔA to both sides and use cancellation. Show this step

$X = B \,\Delta\, A$

Compute both one-sided differences. Show this step

$B \setminus A = \{1, 8\}$ and $A \setminus B = \{2, 7\}$

Unite the exclusive parts and verify by substitution.

$X = \{1, 2, 7, 8\}; \{1, 2, 7, 8\} \,\Delta\, \{2, 5, 7\} = \{1, 5, 8\}$

Answer

A — $\{1, 2, 7, 8\}$

Substitution returns $B = \{1, 5, 8\}$.

Watch Out

Complementing each part but keeping the original connective.

It is tempting to change only each set, but the complement changes the membership condition: 'in at least one' becomes 'in all of the complements', and vice versa.

Complement every operand and switch union with intersection, preserving brackets and the fixed universal set. For symmetric difference, do not stop at $A \setminus B$; also include $B \setminus A$.

Check Yourself

Medium

For sets P, Q, R, which expression is equivalent to $((P \cap Q)^c \cup R)^c$?

A $P \cap Q \cap R^c$

B $P^c \cup Q^c \cup R^c$

C $P^c \cap Q^c \cap R$

D $P \cup Q \cup R^c$

E $P \cap Q \cap R$

Answer

A — $P \cap Q \cap R^c$

Apply De Morgan first to the outer union, then cancel the two complements on $P \cap Q$.

Medium

Let $A = \{1, 4, 6, 9\}$ and $B = \{4, 5, 9, 10\}$. If $X \,\Delta\, A = B$, which set is X?

A $\{1, 5, 6, 10\}$

B $\{4, 6, 9\}$

C $\{1, 4, 5, 6, 9, 10\}$

D $\{5, 10\}$

E $\{1, 4, 6\}$

Answer

A — $\{1, 5, 6, 10\}$

Use $X = B \,\Delta\, A$, retaining elements in exactly one of the two given sets.

Master

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $M = \{1, 3, 6\}$ and $N = \{3, 4, 8\}$. If $Y \,\Delta\, \{3, 8\} = (M \cup N)^c$, which option gives Y?

A $\{2, 3, 5, 7, 8, 9\}$

B $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

D $\{1, 3, 4, 6, 8\}$

E $\{3, 8\}$

Answer

A — $\{2, 3, 5, 7, 8, 9\}$

First complement the union relative to U, then apply symmetric-difference cancellation with $\{3, 8\}$.